DIRECTIONAL FUNCTIONS FOR ORIENTATION DISTRIBUTION ESTIMATION

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ABSTRACT

We propose a novel method to obtain the orientation distribution function (ODF) from diffusion weighted signals measured using the Q-ball imaging methodology. Past work has involved using the spherical harmonics or radial basis functions to represent the ODF. In this work, we propose a novel function that has the ability to compactly represent signals measured using high angular resolution diffusion imaging (HARDI). Further, the closed form solution for computing the corresponding ODF turns out to be the Watson distribution used in directional statistics. As a special case, we formulate a directional function for which the ODF is the von Mises-Fisher (vMF) distribution. Experiments on some synthetic data show the robustness of the proposed method to represent ODF's in the presence of noise. We also compare the method with ODF estimation using spherical harmonics and show some of its advantages.

Index Terms**—** Orientation distribution function (ODF), Directional functions, HARDI, Q-Ball Imaging

1. INTRODUCTION

High angular resolution diffusion imaging (HARDI) has become an important tool in the analysis of brain matter. It essentially allows to compute the probabilities for the displacement of water molecules in tissue fibers. While diffusion tensor imaging (DTI) also allows to capture similar characteristics, it has a significant limitation, in that the technique can only resolve a single fiber direction within each voxel. This shortcoming is significant since human cerebral white matter possesses considerable intravoxel structure at the millimeter resolution typical of MRI. Thus, the single tensor model is inadequate for resolving neural architecture in regions with complex fiber patterns. Further, in regions of fiber crossings, the interpretation of tensor anisotropy becomes complicated [1]. An alternative technique, called q-space imaging (QSI) was proposed [2, 3] which sampled the diffusion signal on a 3D Cartesian lattice. This method addressed the problems plaguing DTI, but required unacceptably long time durations to perform the sampling.

To alleviate these problems, a different method was proposed based on sampling on a spherical shell in diffusion wavevector space. This spherical sampling approach is referred to as HARDI in the literature [2, 4]. In the seminal work by Tuch [1], the author proposed a novel way to compute the orientation distribution function (ODF) by using the Funk-Radon transform (special case of spherical radon transform). The reconstruction of ODF from the measured HARDI signals (defined on the sphere) is referred to as Q-ball imaging in popular literature in the medical community. The ODF in a particular direction is computed by integrating the measured signal along the corresponding equator [1].

1.1. Related Work

Most of the current research is now focused on efficient and accurate computation of the ODF profiles from the measured signal. Apart from Tuch [1], numerous methods have been proposed. Specifically, Jansons *e*t.al. [5] solves the problem in the Fourier domain using a maximum entropy parametrization, although their numerical approximations are sensitive to local minima and are not in general guaranteed to converge to global minima. Peled *e*t.al. [6] use the coordinate frame of a single tensor fit to estimate 2 tensors residing in the resulting plane. It is not however known if the method can be extended to estimate more than 2 tensors. Recent work by Bergmann *e*t.al.[7] provides a way to estimate the ODF profile by weighting each signal inversely with the distance from the equator. They also use a binary integer programming approach to estimate the different tensors from the computed ODF. Recently, spherical harmonics were used in [8, 9, 10] to estimate the ODF. This method works by first computing the coefficients of the spherical harmonic basis of order l that best fit the measured signal and subsequent modification of the coefficients to obtain the desired ODF. In general, it requires $\frac{(l+1)(l+2)}{2}$ spherical harmonic coefficients to represent an ODF.

1.2. Background

In diffusion weighted imaging, the image contrast is related to the diffusion of water molecules, where the measurements can be made sensitive to water diffusion along n distinct spatial directions $\mathbf{u}_1, ..., \mathbf{u}_n \in \mathbb{S}^2 \subset \mathbb{R}^3$ on the sphere, such that a signal $S_1, ..., S_n$ is obtained for each direction. The model developed in [1, 5] for multiple fibers relates the signals with

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the gradient direction in the following manner:

$$S_i = S_0 \sum_{j=1}^{m} w_j \exp(-b\mathbf{u}_i^T D_j \mathbf{u}_i) \tag{1}$$

where w_j are non-negative weights summing upto 1, b is an acquisition specific constant, S_0 is signal intensity without diffusion sensitization and $\mathbf{D}_j$ is the j^{th} diffusion tensor. For synthetic experiments, we will use this model to obtain the signal values S_i and use the analytical formula given in [1] to obtain the corresponding ODF profiles.

2. DIRECTIONAL FUNCTION

For a single tensor model, there are three shapes of the diffusion tensor that any directional function should reasonably approximate, i.e., ellipsoidal, planar and spherical. Any diffusion tensor D can be decomposed as $D = U\Lambda U^T$, where U is a rotation matrix and Λ is a diagonal matrix with eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$. The eigenvalues determine the shape of the tensor, for example, if $\lambda_1 > \lambda_2 > \lambda_3$, the shape is ellipsoidal with the major axis of the ellipsoid pointing to the eigenvector corresponding to λ_1. This is one of the most commonly occurring cases and very useful in tractography and segmentation. Intuitively, this shape represents strong diffusion of water molecules along a particular direction. For $\lambda_1 = \lambda_2 \gg \lambda_3$, the shape is planar indicating diffusion along orthogonal directions and $\lambda_1 = \lambda_2 = \lambda_3$, the diffusion is isotropic (spherical).

For principal diffusion along a particular direction $\mathbf{m}$ (the ellipsoidal case), one can approximate the exponent in the model (1) as $-b\mathbf{u}^T D\mathbf{u} \approx -b\lambda_1\mathbf{u}^T(\mathbf{m}\mathbf{m}^T)\mathbf{u} = -b\lambda_1(\cos(\theta))^2$, where $\theta = \cos^{-1}(\mathbf{m}^T\mathbf{u})$. Thus, for a single tensor, the model (1) can be approximated as

$$\begin{aligned} S_i &= S_0 \exp(-b\lambda_1 \cos^2\theta) = S_0 \exp(-b\lambda_1)\exp(b\lambda_1 \sin^2\theta) \\ &= \tilde{S}_0 \exp(b\lambda_1 \sin^2\theta);\ \theta = \cos^{-1}(\mathbf{u}_i^T\mathbf{m}). \end{aligned}$$

We thus propose to model the measured signal using

$$S_i = A\exp(k\sin^2\theta);\ \theta = \cos^{-1}(\mathbf{u}_i^T\mathbf{m}), \tag{2}$$

where A is a normalizing constant, $\mathbf{m}$ is the principal direction and k is the concentration parameter that determines how anisotropic the diffusion is. In our subsequent discussions we will ignore the normalizing constant A, since one can always normalize the signal using the min-max normalization [1] so that the range of S lies between 0 and 1. This directional function (2) can also represent other types of diffusion that a single tensor is capable of representing. Thus, near isotropic diffusion is obtained by setting $k \to 0$, while ellipsoidal diffusion (as discussed above) is given by $k > 0$. Planar diffusion can be obtained by setting $k < 0$. Ideally, if one models the diffusion with an ODF, then a planar tensor shape can be better represented with a linear combination of 2 directional functions orthogonal to each other. Figure 1 shows the signal surface using the proposed function for 3 different values of k.

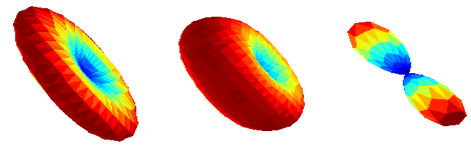

Fig. 1. The surface $S_i\mathbf{u}_i$ for $k = 5, 1e^{-10}, -5$ respectively (left to right).

Thus, the proposed function can be used as an alternative to the usual tensor representation in DTI images. In particular, note that a 3D tensor requires 6 parameters to describe it, while the proposed function requires only 3 free parameters (2 for representing the direction in spherical coordinates and 1 for the concentration parameter k). The value of k can be used to compute important measures of clinical relevance like fractional anisotropy (FA). One such formula is

$$FA = 1 - \exp(- \mid k \mid).$$

Further, one can also perform interpolation and compute distances between 2 directional functions in a straightforward manner. However, in the interest of space, we will not discuss it in this work.

2.1. The Orientation Distribution Function

The ODF of the function (2) can be computed in a straightforward manner. As given in [1], the ODF is obtained by integrating over the radial component of the measured signal S. Thus, the ODF at a direction $\mathbf{u}_i$ is obtained by integrating the signal values along the corresponding equator. From simple trigonometry, it is clear that, given a diffusion direction $\mathbf{m}$, the function (2) represents an "equatorial girdle" along the corresponding equator (see Figure 1) due to the action of the sine function. Thus, the corresponding ODF can be directly obtained by replacing the sine function with the cosine function noting that there is a phase difference of $\pi/2$ between them. Thus, the ODF is given by

$$O_i = B\exp(k\cos^2\theta);\ \theta = \cos^{-1}(\mathbf{u}_i^T\mathbf{m}), \tag{3}$$

where B is a normalizing constant which we will ignore in this work. Figure 2 shows the measured signal and the corresponding ODF for a diffusion direction $\mathbf{m}$. Notice the planar shape of the ODF corresponding to negative k value.

Thus, once the principal diffusion direction $\mathbf{m}$ is known, the ODF can be analytically computed using (3). This function has been used in the field of directional statistics and is known as the Watson density function [11]. We should however note that, this the first time it has been applied in medical imaging and a mathematical justification has been given for its use.

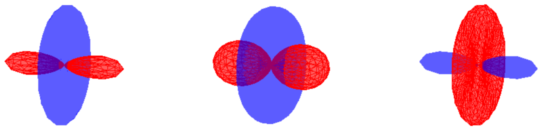

Fig. 2. The signal surface (blue) and the corresponding ODF (red) for $k = 5, 0.0001, -5$ respectively (left to right).

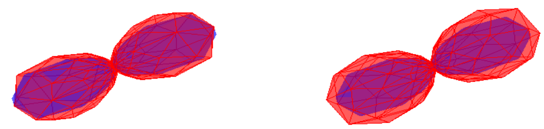

Fig. 3. The true ODF (blue) and the estimated ODF (red). Left: using Watson function, Right: using spherical harmonics of order 8.

Another function that is commonly used to represent directional data is the von Mises-Fisher (vMF) distribution [11, 12]. In particular, the authors in [12] have used a mixture of vMF's to represent multi-fiber orientations. They however have not dealt with the topic of estimation of the ODF from the measured signal. Instead, it is assumed that the ODF has already been computed using any known method and then an EM algorithm is used to fit the mixture of vMF's to the ODF. Just as in the case of Watson density, one can use the following function so that the ODF can be represented by the vMF distribution function. If one assumes that the signal profile can be represented by: $S_i = A\exp(k\sin\theta)$; $\theta = \cos^{-1}(\mathbf{u}_i^T\mathbf{m})$, then, the corresponding ODF can be computed using

$$V_i = B\exp(k\cos\theta) = B\exp(k\mathbf{m}^T\mathbf{u}_i). \qquad (4)$$

Apart from the power of sine and cosine in the exponent, there are other differences between these 2 functions (Watson and vMF). The vMF function cannot represent "equatorial girdles" or planar shapes. Thus, negative values of k do not provide any additional information than positive values, implying $k \in \mathcal{R}^+$. Also, in the case of vMF, one needs to explicitly compute the antipodally symmetric diffusion surface for display purposes. For all other cases, both these functions behave in a similar fashion.

3. EXPERIMENTS WITH 1 TENSOR MODEL

In this section, using some synthetic experiments for a 1 tensor model, we compare the representation ability of the directional function (3) with that of spherical harmonics of a certain order l. In particular, for spherical harmonics, we will use the method given in [8] with the regularizing parameter to compute an ODF. The signals were generated using the model (1) and the true ODF was obtained using the formula given in [1]. We compute the MSE (mean squared error) given by $\left(\frac{\|S-\hat{S}\|}{\|S\|}\right)^2$, where S is the true signal and $\hat{S}$ is the estimated signal. Rician noise was added to obtain a signal with SNR = 10 dB. 81 gradient directions were used.

We first find the best fit of the function (2) to the measured signal S to obtain the parameter values $k, \mathbf{m}$. The corresponding ODF is then given by the Watson function. One can use any method [13, 11] to find the parameters $k, \mathbf{m}$. In this work however, we have done a brute force search to find the parameters that give the least error $\| S - \hat{S} \|$. For the planar case, we used the following values to generate the signal (1): $b = 1000$ and eigenvalues of $1e^{-6}\{4000, 4000, 1000\}$. 1000 samples were generated with random orientation. The MSE between the true signal and estimated signal was 0.1183, while MSE between the corresponding ODF was 0.0242. The MSE for the estimated ODF using spherical harmonics of order 8 was 0.0034.

For the ellipsoidal case, the signal was generated with eigenvalues of $1e^{-6}\{1700, 300, 300\}$ with the same b-value as earlier. For 1000 random samples, MSE for signal estimation was 0.0492, for the corresponding ODF was 0.0733 and for estimation using spherical harmonics of order 8 (SH8) was 0.1929. Thus, for the planar case, the spherical harmonics performed better while for ellipsoidal case the Watson function had lower error. Figure 3 shows the ODF estimated using the Watson function and spherical harmonics. With the same set of parameters, errors (for 1000 random samples) using the vMF distribution were: MSE (signal) = 0.0495, MSE(ODF) = 0.0765, MSE(SH8) = 0.1902.

4. MIXTURE MODELS

In order to represent multiple fiber orientations, a mixture model can be used. Thus, the measured signal can approximated using a mixture of m directional functions:

$$S_i = \sum_{j=1}^{m} c_j \exp(k_j \sin^2\theta_j);\ \theta_j = \cos^{-1}(\mathbf{m}_j^T\mathbf{u}_i).$$

By linearity, the corresponding ODF is given by

$$O_i = \sum_{j=1}^{m} c_j \exp(k_j \cos^2\theta_j). \qquad (5)$$

Thus, to represent multiple fibers, it requires 4 parameters per fiber direction. Estimation of the mixture parameters is itself a topic of active research. One can use the EM-style algorithm given in [13] to faithfully represent the measured signal S. The computation of the corresponding ODF is then straightforward. The same procedure can be applied for the vMF distribution function as well. Figure 4 shows the signal and the corresponding ODF computed using the Watson function for 2 fiber orientations. Figure 5 shows the ODF obtained

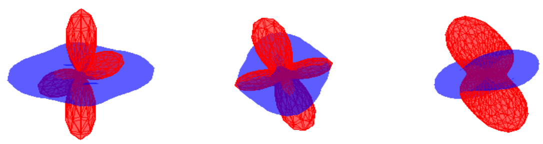

Fig. 4. The signal (blue) and the estimated ODF (red) for 2 fiber orientations.

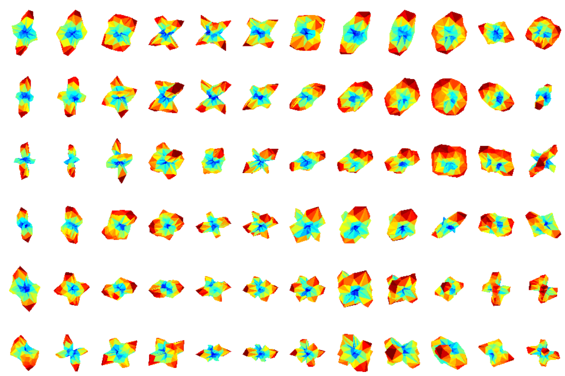

Fig. 5. ODF estimation on real data set with 55 gradient directions.

using the Watson function on a real data set with 55 gradient directions.

As discussed earlier, with the proposed directional functions only $2m$ parameters are required for representing m fiber orientations quite accurately. This is in contrast to spherical harmonics, where an order 8 expansion requires 45 coefficients. However, the computation of these coefficients is quite fast [8] and robust to noise even for sparse sampling of the sphere. This latter fact can be utilized for preprocessing of noisy data and the directional functions can then be fitted to the smoothed data using any method (like, [13]) to obtain a very compact representation of the ODF. Further, for the vMF distributions, the authors in [12] have described algorithms for interpolating and computing geodesic distances between 2 ODF's. Since the parameter space for the Watson distribution and the vMF distribution is the same, one can directly use the formulation given therein. Another important feature that the present work allows is to perform interpolation on the signal data itself based on the Riemannian formulation given in [12].

5. CONCLUSION

We have introduced a novel directional function for which a closed form expression for computing the corresponding ODF exists. The resulting ODF's, known in the literature as Watson distribution and von Mises-Fisher distribution, provide a representation that requires fewer parameters than spherical harmonics. In particular, the Watson distribution can be used as an alternative representation for a diffusion tensor but with fewer parameters. Mixture models of these functions can be used to efficiently represent multi-fiber orientations. Future work entails coming up with a fast and robust algorithm to determine the parameters of the mixture model.

6. REFERENCES

[1] David S. Tuch, "Q-ball imaging," *Magnetic Resonance in Medicine*, vol. 52, pp. 1358–1372, 2004.

[2] D Tuch, T Reese, M Wiegella, N Makris, J Belliveau, and V Wedeen, "High angular resolution diffusion imaging reveals intravoxel white matter fiber heterogeneity," *Magnetic Resonance in Medicine*, vol. 48, pp. 577–582, 2002.

[3] A Alexander, K Hasan, M Lazar, J Tsuruda, and D Parker, "Analysis of partial volume effects in diffusion-tensor mri," *Magnetic Resonance in Medicine*, vol. 45, pp. 770–780, 2001.

[4] D Alexander, G Barker, and S Arridge, "Detection and modeling of nongaussian apparent diffusion coefficient profiles in human brain data," *Magnetic Resonance in Medicine*, vol. 48, pp. 331–340, 2002.

[5] K Jansons and D Alexander, "Persistent angular structure: New insights from diffusion magnetic resonance imaging data," *Inverse Problems*, vol. 19, pp. 1031–1046, 2003.

[6] S Peled and C-F Westin, "Geometric extraction of two crossing tracts in dwi," in *Proc. of 13th ISMRM*, 2005.

[7] O Bergmann, G. Kindlmann, A. Lundervold, and C-F Westin, "Diffusion k-tensor estimation from q-ball imaging using discretized principal axes," in *MICCAI*, 2006, pp. 268–275.

[8] M. Descoteaux, E. Angelino, S. Fitzgibbons, and R. Deriche, "Regularized, fast and robust analytical q-ball imaging," *Magnetic Resonance in Medicine*, 2007.

[9] E Ozarslan and T. Mareci, "Generalized diffusion tensor imaging and analytical relationships between diffusion tensor imaging and high angular resolution imaging," *Magnetic Resonance in Medicine*, vol. 50, pp. 955–965, 2003.

[10] Y. Chen, W Guo, Q Zeng, X Yan, F Huang, G He, B Vemuri, and Y Liu, "Estimation, smoothing and characterization of apparent diffusion coefficient profiles from high angular resolution DWI," in *CVPR*, 2004, pp. 588–593.

[11] P Jupp and K Mardia, "A unified view of the theory of Directional Statistics," *International Statistical Review*, vol. 57, pp. 261–294, 1989.

[12] T McGraw, B Vemuri, B Yezierski, and T Mareci, "Von mises-fisher mixture model of the diffusion ODF," in *ISBI*, 2006.

[13] A Banerjee, I Dhillon, J Gosh, and S Sra, "Clustering on the unit hypersphere using von Mises-Fisher distributions," *Journal of Machine learning Research*, vol. 6, pp. 1345–1382, 2005.