A VARIABLE PROJECTION APPROACH TO PARALLEL MAGNETIC RESONANCE IMAGING

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ABSTRACT

Accurate estimation of channel sensitivity functions is still a challenging problem in parallel imaging. JSENSE has recently been proposed to improve the accuracy of sensitivity estimation using the self calibration data. It regards both the coil sensitivities and the desired images as unknowns to be solved for jointly. The existing algorithm for the underlying nonlinear optimization problem requires an accurate initial value, which needs considerable number of self calibration data. In this paper, we use the variable projection method to find the optimal solution. The method is known to be able to give an optimal solution, and our implementation has a linear convergence rate. The performance of the proposed method is evaluated using a set of in vivo experiment data.

***Index Terms*—**SENSE, self calibration, separable least squares, variable projection

1. INTRODUCTION

Parallel magnetic resonance imaging (pMRI), as a fast imaging method, uses an array of RF receiver surface coils to acquire multiple sets of under-sampled *k*-space data simultaneously. Over the past few years, a number of pMRI techniques have been proposed for reconstructing a complete MR image from these under-sampled data in either *k*-space or the image-domain. Among these methods, some do not need the explicit functions of coil sensitivity, such as PILS [1], AUTO-SMASH [2], VD-AUTO-SMASH [3], and GRAPPA [4], while others require the functions to be given exactly, such as SMASH [5], SENSE [6], and SPACE-RIP [7]. For the methods in the latter category, the sensitivity estimation method is as important as the reconstruction algorithm [8].

Unfortunately, the existing techniques for determination of sensitivity functions are not yet satisfying. The most common technique has been to derive sensitivities directly from a set of reference images obtained in a separate calibration scan before or after the accelerated scans. This calibration scan can prolong total imaging time, partially counteracting the benefits of reduced acquisition time associated with pMRI. Another practical problem with this technique is that misregistrations or inconsistencies between the calibration scan and the accelerated scan result in artifacts in the reconstructed images, which is a major concern in dynamic imaging applications. Adaptive sensitivity estimation [9,10] have been proposed for these applications. Based solely on the data from accelerated scans, the method uses UNFOLD [11] to generate low-temporal-resolution, aliasing-free reference images for sensitivity estimation. However, UNFOLD is limited to dynamic applications where at least half of the field-of-view remains static over time. A more general method is the self-calibrating technique, which also eliminates a separate calibration scan but acquires a variable density (VD) k-space data during the accelerated scan [8]. The VD acquisition includes a small number of fully-sampled lines at the center of *k*-space, known as auto-calibration signal (ACS) lines, in addition to the down-sampled lines at outer k-space. These central *k*-space lines after Fourier transformation produce low-resolution in vivo reference images. To derive the sensitivities, these low-resolution reference images are divided by their sum-of-squares combination [8,12]. If a small number of ACS lines are used, the truncation errors will be present in all sensitivity functions, which become serious especially at locations where the object transverse magnetization has high spatial frequency components. Consequently, the pMRI reconstruction suffers from aliasing artifacts. Therefore, to improve the sensitivity accuracy with a small number of ACS data is crucial for pMRI techniques to achieve a high acceleration.

JSENSE [13] has recently been proposed to improve the accuracy of sensitivity estimation using the self calibration data. It regards both the coil sensitivities and the desired images as unknowns to be solved for jointly and formulates the reconstruction as a nonlinear problem. The existing algorithm solves the problem by iterative alternating minimization, which requires considerable number of self calibration data for an accurate initial sensitivity estimation.

In this paper, we propose to use the variable projection method [14] to solve the nonlinear optimization problem. The method takes advantage of the separability of two unknown variables to achieve an optimal solution. It requires very few self calibration data because it converges regardless of the initial value. The proposed method has been tested on a set of real data and demonstrated promising results.

2. PROPOSED METHOD

2.1. Problem Formulation

JSENSE is based on VD acquisition with both the ACS data and the reduced data used for reconstruction. In JSENSE, the encoding matrix **E** is formulated as a function of sensitivity, and the imaging equation is given by

$$\mathbf{E}(\mathbf{a})\mathbf{f} = \mathbf{d}\,, \tag{1}$$

where $\mathbf{a}$ is the parameter representing the coil sensitivities and is also an unknown to be solved. Although there can be different ways to parameterize the sensitivities, we use a simple polynomial function to model the coil sensitivities as

$$s_l(\vec{r}) = \sum_{i=0}^{N}\sum_{j=0}^{N} a_{l,i,j}(x-\bar{x})^i(y-\bar{y})^j\,, \tag{2}$$

where $(x,y)=\vec{r}$ denotes the location of a pixel, ($\bar{x}$, $\bar{y}$) denotes the averaged location, and $a_{l,i,j}$ is the coefficient of a polynomial, forming the unknown vector $\mathbf{a}$. We choose the highest power of x and y to be the same and define it as the order of the polynomial. High order polynomial improves the accuracy of the model, but also increases the number of unknowns to be solved for. Because of the smooth nature of coil sensitivity in general, polynomial of low order is usually sufficient. Under this model, the encoding matrix explicitly becomes

$$\mathbf{E}(\mathbf{a})_{\{l,m\},n} = \sum_{i,j} a_{l,i,j}(x_n-\bar{x})^i(y_n-\bar{y})^j e^{-i2\pi(k_{xm}\cdot x_n + k_{ym}\cdot y_n)} \tag{3}$$

Taking into account *k*-space data noise which is usually additive white Gaussian, we can jointly estimate the coefficients for coil sensitivities $\mathbf{a}$ and the desired image $\mathbf{f}$ by finding a penalized least-squares solution. Specifically,

$$\{\mathbf{a},\mathbf{f}\} = \arg\min_{\{\mathbf{a},\mathbf{f}\}} U(\mathbf{a},\mathbf{f})\,, \tag{4}$$

where the cost function to be minimized is

$$U(\mathbf{a},\mathbf{f}) = \frac{1}{2}\|\mathbf{e}\|^2 = \frac{1}{2}\|\mathbf{E}(\mathbf{a})\mathbf{f}-\mathbf{d}\|^2\,. \tag{5}$$

2.2. Variable Projection Method

The most straightforward way to obtain a numerical solution of the above nonlinear least squares problem in Eq. (4) is the Newton algorithm. The update equation for the *k*th iteration is given by

$$\begin{bmatrix}\mathbf{f}_{k+1}\\ \mathbf{a}_{k+1}\end{bmatrix} = \begin{bmatrix}\mathbf{f}_{k}\\ \mathbf{a}_{k}\end{bmatrix} - (U_k'')^{-1}U_k'\,, \tag{6}$$

where the gradient and the Hessian are given by

$$U' = \begin{bmatrix}\mathbf{e}_{\mathbf{f}}^H\mathbf{e}\\ \mathbf{e}_{\mathbf{a}}^H\mathbf{e}\end{bmatrix} \tag{7}$$

and

$$U'' = \begin{bmatrix}\mathbf{e}_{\mathbf{f}}^H\mathbf{e}_{\mathbf{f}} + \mathbf{e}_{\mathbf{ff}}^H\mathbf{e} & \mathbf{e}_{\mathbf{f}}^H\mathbf{e}_{\mathbf{a}} + \mathbf{e}_{\mathbf{af}}^H\mathbf{e}\\ \mathbf{e}_{\mathbf{a}}^H\mathbf{e}_{\mathbf{f}} + \mathbf{e}_{\mathbf{af}}^H\mathbf{e} & \mathbf{e}_{\mathbf{a}}^H\mathbf{e}_{\mathbf{a}} + \mathbf{e}_{\mathbf{aa}}^H\mathbf{e}\end{bmatrix}. \tag{8}$$

where matrices $\mathbf{e}_{\mathbf{f}}$ and $\mathbf{e}_{\mathbf{a}}$ denote the partial derivatives of the error vector **e** with respect to the vector **f** and **a,** $\mathbf{e}_{\mathbf{aa}}, \mathbf{e}_{\mathbf{af}}$, and $\mathbf{e}_{\mathbf{ff}}$ denote the second order derivatives with respect to **(a,a)**, **(a,f)**, and **(f,f)**, respectively. However, computation of gradient and Hessian is intensive due to the large size of the problem. To efficiently find the optimal solution to (4), we take advantage of the fact that the two unknowns **a** and **f** are separable and use the variable projection (VP) method [14] for the nonlinear least squares problem. Specifically, the conventional SENSE solution

$$\mathbf{f} = [\mathbf{E}^H(\mathbf{a})\mathbf{E}(\mathbf{a})]^{-1}\mathbf{E}^H(\mathbf{a})\mathbf{d} \tag{9}$$

is plugged into Eq. (1), and the optimization problem in Eq. (4) is simplified to

$$\mathbf{a} = \arg\min_{\mathbf{a}}\left\|\mathbf{E}(\mathbf{a})[\mathbf{E}^H(\mathbf{a})\mathbf{E}(\mathbf{a})]^{-1}\mathbf{E}^H(\mathbf{a})\mathbf{d} - \mathbf{d}\right\|_2^2 \tag{10}$$

where only the parameters of the sensitivity functions $\mathbf{a}$ are to be sought for. Given $\mathbf{a}$, the desired image $\mathbf{f}$ can then be reconstructed by Eq. (9). It was proved [14] that

(i) If **a** is a critical point (or a global minimizer) of Eq. (10), and **f** is obtained by Eq. (9) accordingly, then **(a,f)** is a critical point (or a global minimizer) of Eq. (4).

(ii) If **(a,f)** is a global minimizer of Eq. (4), then **a** is a global minimizer of Eq. (10) and the residual of both functions are equal. Furthermore, if there is a unique **f** among the minimizing pairs in Eq. (4), then **f** must satisfy Eq. (4).

Therefore, the VP solution is the same as the solution to the original problem in Eq. (4).

Applying the Newton algorithm to the separated problem, the gradient and Hessian matrix are given by

$$U_{\mathbf{a}}' = \mathbf{e}_{\mathbf{a}}^H\mathbf{e} \tag{11}$$

and

$$\begin{aligned}U_{\mathbf{a}}'' = {} & \mathbf{e}_{\mathbf{a}}^H\mathbf{e}_{\mathbf{a}} + \mathbf{e}_{\mathbf{aa}}^H\mathbf{e} - \\ & (\mathbf{e}_{\mathbf{f}}^H\mathbf{e}_{\mathbf{a}} + \mathbf{e}_{\mathbf{af}}^H\mathbf{e})^H(\mathbf{e}_{\mathbf{f}}^H\mathbf{e}_{\mathbf{f}} + \mathbf{e}_{\mathbf{ff}}^H\mathbf{e})^{-1}(\mathbf{e}_{\mathbf{f}}^H\mathbf{e}_{\mathbf{a}} + \mathbf{e}_{\mathbf{af}}^H\mathbf{e})\end{aligned} \tag{12}$$

in the Newton algorithm. For the JSENSE formulation, we have

$$\mathbf{e}_{\mathbf{f}} = \mathbf{E}(\mathbf{a})$$
$$\mathbf{e}_{\mathbf{a}} = \mathbf{E}'(\mathbf{a})\mathbf{f}$$
$$\mathbf{e}_{\mathbf{aa}} = \mathbf{0}$$
$$\mathbf{e}_{\mathbf{af}} = [\mathbf{E}'(\mathbf{a})]$$
$$\mathbf{e}_{\mathbf{ff}} = \mathbf{0}$$

Noting that $\mathbf{e}_{\mathbf{ff}} = 0$ and $\mathbf{e}_{\mathbf{aa}} = 0$ because **e** is a linear function of both **f** and **a**, the Hessian can be computed without evaluating second derivatives. However, the

Hessian cannot be assumed to be positive definite. To simplify computation, the variable change is ignored in the separated problem, and **f** is kept fixed when solving for **a** in each step. Then the Hessian is approximated as

$$U_{\mathbf{a}}'' = \mathbf{e}_{\mathbf{a}}^H \mathbf{e}_{\mathbf{a}} \tag{13}$$

Thereby, the update equation becomes

$$\mathbf{a}_{k+1} = \mathbf{a}_k - [(\mathbf{E}_k'\mathbf{f}_k)^+ \mathbf{Q}_{\mathbf{E}_k}\mathbf{d}] \tag{14}$$

where $\mathbf{E}_k = \mathbf{E}(\mathbf{a}_k)$, $\mathbf{Q}_{\mathbf{E}_k}\mathbf{d} = (\mathbf{d} - \mathbf{E}_k\mathbf{f}_k)$, and $^+$ denotes the pseudo inverse. It has been shown that the algorithm has a linear convergence rate, which is higher than that of the Gauss algorithm applied to the nonseparated problem [15].

2.3. Efficient Implementation

Due to the large size of matrix $\mathbf{E}_k'\mathbf{f}_k$, calculation of the pseudo inverse in Eq. (14) can be computationally intensive. QR factorization is usually used for reduction of computational complexity. In this case, QR factorization requires explicit representation of the large matrix $\mathbf{E}_k'$, which is memory demanding. Instead of QR factorization, we use iterative conjugated gradient method for calculation of the pseudo inverse where all matrix multiplications are implemented by direct operations. Detailed description of the VP algorithm is as follows.

<u>VP Algorithm</u>:

Initialize $\mathbf{a}^0$**;**

For k=0, 1, …

  Compute $\mathbf{f}_k = [\mathbf{E}^H(\mathbf{a}_k)\mathbf{E}(\mathbf{a}_k)]^{-1}\mathbf{E}^H(\mathbf{a}_k)\mathbf{d}$;

  Test for convergence;

  Calculate $(\mathbf{E}_k'\mathbf{f}_k)$ by taking the derivative of the fast Fourier transform of the sensitivity weighted $\mathbf{f}_k$

  Calculate $(\mathbf{E}_k'\mathbf{f}_k)^+$ using iterative conjugate gradient

  Calculate the fitting error $\mathbf{Q}_{\mathbf{E}_k}\mathbf{d} = (\mathbf{d} - \mathbf{E}_k\mathbf{f}_k)$

  Increment $\mathbf{a}_{k+1} = \mathbf{a}_k - [(\mathbf{E}_k'\mathbf{f}_k)^+ \mathbf{Q}_{\mathbf{E}_k}\mathbf{d}]$**;**

end

  Sensitivity parameter $\mathbf{a}=\mathbf{a}^{k+1}$;

  Reconstructed image $\mathbf{f} = [\mathbf{E}^H(\mathbf{a})\mathbf{E}(\mathbf{a})]^{-1}\mathbf{E}^H(\mathbf{a})\mathbf{d}$.

The novel nonlinear formulation of image reconstruction allows the sensitivity and the image to be estimated simultaneously, thereby prevents the errors of the independently-estimated sensitivities from propagating to the final reconstruction as in conventional SENSE.

3. EXPERIMENT RESULTS

The proposed approach was tested on a set of real data. The in vivo brain data were acquired on a 3T commercial scanner (GE Healthcare, Waukesha, WI) and an 8-channel head coil (Invivo, Gainesville, FL) was used to scan a healthy volunteer with a 2D T1-weighted spin echo protocol (axial plane, TE/TR = 11/700 ms, 22cm FOV, 10 slices, 256x256 matrix). Informed consent was obtained from the volunteer in accordance with the institutional review board policy. The fully sampled k-space data were used to generate a sum-of-square (SoS) reconstruction, which is used as the reference image for error calculation. We manually remove some phase encoding lines to simulate a reduction factor of 4 in outer k-space, and the number of ACS lines in central k-space of 16, 24, and 32. The performance of the proposed algorithm for the VD acquisition can be evaluated visually in Fig. 1.

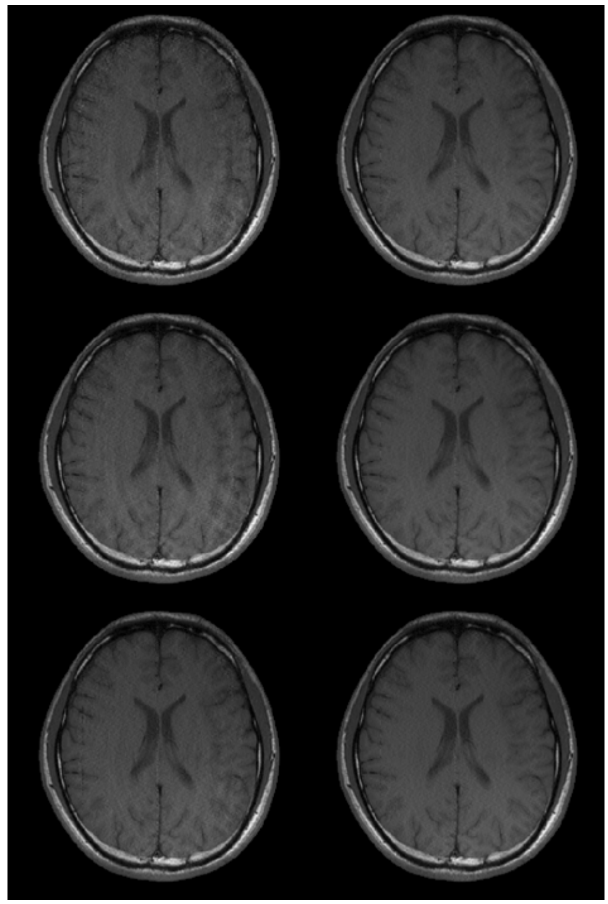

Fig. 1 Reconstructed images using the self-calibrated SENSE (left) and the VP reconstruction (right). The top, middle, and bottom rows are reconstructions using 16, 24, and 32 ACS lines, respectively.

It is seen that the proposed VP method greatly reduces the image aliasing artifacts in SENSE reconstruction using the conjugate gradient (CG) method [5] with self-calibrated sensitivities. The improvement is especially significant when less ACS lines are used. In addition, the VP reconstructions look quite similar for difference ACS lines except that the signal-to-noise ratios are slightly improved due to the increased number of data used in reconstruction. It suggests the VP method barely depends on the initial values as long as they are close to the optimal values.

The convergence behavior of the VP method has also been investigated. Figure 2 compares the normalized mean squared errors as a function of the number of iterations for different ACS lines.

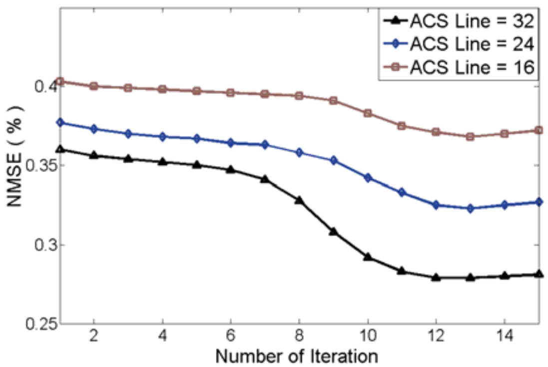

Fig. 2 The convergence behavior of the proposed VP method for different number of ACS lines.

5. DISCUSSION AND CONCLUSION

The proposed method uses VP to solve the underlying optimization problem in JSENSE, and thereby significantly reduces the number of self calibration data needed during the accelerated scans. Although the non-convexity of the JSENSE formulation might result in multiple local optimal solutions which are dependent on the initial value, our experiments indicate the estimated low-resolution sensitivity is a good starting point to converge to the desired global optimal solution. To address the intensive memory and computation demands, we have also developed efficient implementation using pure operations directly without the explicit representation of the large encoding matrix. The method should prove useful for further reduction of scan time in many fast imaging applications.

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